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A Class of Nonlinear Averaging Integral Operators

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Let $\mathbf{H}$ be the class of analytic functions defined in the unit disc U, and let co E denote the convex hull of a set E in $\mathbf{C}$. If $\mathbf{K} \subset \mathbf{H}$, then an operator $I: \mathbf{K} \to \mathbf{H}$ is an averaging operator if $I[f](0) = f(0)$ and $I[f](U) \subset \operatorname{co} f(U)$, for all $f \in \mathbf{K}$. The authors show that the operator $I_{\beta,\gamma}[f](z) \equiv [\gamma z^{-\gamma} \int_0^z f^\beta(t) t^{\gamma-1}\, dt]^{1/\beta}$ is an averaging operator on certain subsets of $\mathbf{H}$.

1. INTRODUCTION

Let $\mathbf{H}$ be the class of analytic functions defined in the unit disc U and let co E denote the convex hull of a set E in $\mathbf{C}$. In [2] the authors introduced the concept of an *averaging operator defined on a set* $\mathbf{K} \subset \mathbf{H}$. This is an operator $I: \mathbf{K} \to \mathbf{H}$ that satisfies $I[f](0) = f(0)$ and

$$I[f](U) \subset \operatorname{co} f(U), \tag{1}$$

for all $f \in \mathbf{K}$. Also in [2] the authors determined conditions for, and gave

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examples of, linear integral operators that are also averaging operators. One such example is the operator

$$I_\gamma[f](z) \equiv \gamma z^{-\gamma} \int_0^z f(t) t^{\gamma-1}\, dt, \tag{2}$$

where $\operatorname{Re} \gamma > 0$; it is shown [2, Example 2] that I_γ satisfies (1) for $f \in \mathbf{H}$ and hence is an averaging operator on $\mathbf{H}$.

In this short note we describe a class of nonlinear integral operators that are also averaging operators. The operators are a generalization of the operator given in (2) and are of the form

$$I_{\beta,\gamma}[f](z) \equiv \left[\gamma z^{-\gamma} \int_0^z f^\beta(t) t^{\gamma-1}\, dt \right]^{1/\beta}. \tag{3}$$

In order to show that these operators are averaging operators on certain subsets of $\mathbf{H}$ we need to employ a different proof technique from that used in the aforementioned linear case. The proof depends on several lemmas involving differential subordinations plus a subordination chain argument.

II. Preliminaries

Since subordination is crucial to our proof, we first restate its definition (in a restricted sense). Let f and F be analytic in U. The function f *is subordinate to* F, written $f \prec F$ or $f(z) \prec F(z)$, if F is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

We also need the definition of a *convex* function; this is a function $f \in \mathbf{H}$ which is univalent and for which $f(U)$ is a convex domain.

Lemma 1. *Let p be analytic in U and let h be convex on $\overline{U}$, with $p(0) = h(0)$. If p is not subordinate to h, then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$, and an $m \geq 1$ for which $p(|z| < |z_0|) \subset h(U)$,*

(i) $p(z_0) = h(\zeta_0)$ *and*

(ii) $z_0 p'(z_0) = m \zeta_0 h'(\zeta_0)$.

A more general form of this result appears as Lemma 1 in [1].

Lemma 2 [2, Lemma 2]. *Let $\mathbf{K} \subset \mathbf{H}$ and let an operator $I : \mathbf{K} \to \mathbf{H}$ satisfy $I[f](0) = f(0)$ for all $f \in \mathbf{K}$. A necessary and sufficient condition for I to be an averaging operator on $\mathbf{K}$ is that*

$$(f \in \mathbf{K}, h \text{ convex, and } f \prec h) \Rightarrow I[f] \prec h.$$

The next lemma concerns subordination (or Loewner) chains. A function $L(z, t)$, $z \in U$, $t \geq 0$ is a *subordination chain* if $L(\cdot, t)$ is analytic and univalent in U for all $t \geq 0$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and $L(z, s) \prec L(z, t)$ when $0 \leq s \leq t$ [4, p. 157].

LEMMA 3 [4, p. 159]. *The function* $L(z, t) = a_1(t)z + \cdots$, *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if*

$$\operatorname{Re}\left[z\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right] > 0, \tag{4}$$

for $z \in U$ *and* $t \geq 0$.

Before introducing the final lemma, we need to consider a special mapping from U onto a slit domain. This mapping plays a crucial role in both the lemma and the main theorem of the article.

Let $c \in \mathbf{C}$ with $\operatorname{Re} c > 0$ and let

$$N = N(c) = [|c|(1 + 2\operatorname{Re} c)^{1/2} + \operatorname{Im} c]/\operatorname{Re} c.$$

If h is the univalent function $h(z) = 2Nz/(1 - z^2)$ and $b \equiv h^{-1}(c)$ then let

$$Q_c(z) \equiv h[(z + b)/(1 + \overline{b}z)] = 2N\frac{(z + b)(1 + \overline{b}z)}{(1 + \overline{b}z)^2 - (z + b)^2}, \tag{5}$$

$z \in U$. The function Q_c is univalent in U, $Q_c(0) = c$, and $Q_c(U) = h(U)$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geq N$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leq -N$.

LEMMA 4 [3, Theorem 2]. *Let* $\beta, \gamma \in \mathbf{C}$ *with* $\beta \neq 0$ *and* $\operatorname{Re}(\beta + \gamma) > 0$, *and let* $f \in \mathbf{H}$ *be normalized so that* $f(0) = 0$ *and* $f'(0) \neq 0$. *If*

$$\beta z f'(z)/f(z) + \gamma \prec Q_{\beta+\gamma}(z),$$

where Q_c *is defined by* (5), *then the function*

$$G(z) \equiv J_{\beta,\gamma}[f](z) \equiv \left[\frac{\beta + \gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}\,dt\right]^{1/\beta} \tag{6}$$

is analytic in U, $G(z)/z \neq 0$, *and* $\operatorname{Re}[\beta z G'(z)/G(z) + \gamma] > 0$.

III. Main Results

Before we establish our main result we need to define the subsets of $\mathbf{H}$ on which the integral operator given by (3) will be defined. Let $\beta \geq 1$, let $\gamma \in \mathbf{C}$ with $\operatorname{Re}\gamma > 0$ and let

$$\mathbf{E}_{\beta,\gamma} \equiv \begin{cases} \left\{ f \in \mathbf{H} : f(0) = 0, f'(0) \neq 0, \beta \dfrac{zf'(z)}{f(z)} + \gamma \prec Q_{\beta+\gamma} \right\}, & \beta > 1, \\ \mathbf{H}, & \beta = 1. \end{cases}$$

Theorem 1. *If $\beta \geq 1$ and if $\gamma \in \mathbf{C}$ with* $\operatorname{Re}\gamma > 0$, *then the operator $I_{\beta,\gamma}$ defined by*

$$I_{\beta,\gamma}[f](z) = \left[\gamma z^{-\gamma} \int_0^z f^{\beta}(t) t^{\gamma-1}\, dt \right]^{1/\beta} \tag{7}$$

is an averaging operator on $\mathbf{E}_{\beta,\gamma}$. (*In* (7) *all powers are principal values*.)

Proof. For $\beta = 1$, the operator given by (7) reduces to the operator given by (2), which is an averaging operator on $\mathbf{E}_{1,\gamma} = \mathbf{H}$.

Suppose $\beta > 1$ and let $f \in \mathbf{E}_{\beta,\gamma}$. We will use all four of the lemmas of Section 2 to prove this case. According to Lemma 2, we can show that $I_{\beta,\gamma}$ is an averaging operator by showing that if h is convex and $f \prec h$ then $I_{\beta,\gamma}[f] \prec h$. So we assume that f is subordinate to a convex function h.

In addition, we will assume that h is convex on $\overline{U}$. If not we can define $h_r(z) = h(rz)$ and $I_{\beta,\gamma}[f_r](z) \equiv I_{\beta,\gamma}[f](rz)$. The function h_r is convex on $\overline{U}$ and the proof that follows will show that $I_{\beta,\gamma}[f_r] \prec h_r$. By letting $r \to 1^-$ we obtain $I_{\beta,\gamma}[f] \prec h$.

If we let $F = I_{\beta,\gamma}[f]$ then from (7) we obtain

$$F^{\beta}(z) \left[1 + \frac{\beta}{\gamma} \frac{zF'(z)}{F(z)} \right] = f^{\beta}(z). \tag{8}$$

By comparing (6) and (7) we obtain $F(z) = G(z) \cdot [\gamma/(\beta + \gamma)]^{1/\beta}$; hence, $\beta zF'(z)/F(z) + \gamma = \beta zG'(z)/G(z) + \gamma$. Since $f \in \mathbf{E}_{\beta,\gamma}$, from Lemma 4 we obtain $\operatorname{Re}[\beta zF'(z)/F(z) + \gamma] > 0$. Hence $1 + (\beta/\gamma)(zF'(z)/F(z)) \neq 0$ in U and from (8) we obtain

$$F(z) \left[1 + \frac{\beta}{\gamma} \frac{zF'(z)}{F(z)} \right]^{1/\beta} = f(z).$$

We need to show that

$$F(z)\left[1+\frac{\beta}{\gamma}\frac{zF'(z)}{F(z)}\right]^{1/\beta} \prec h(z) \Rightarrow F(z) \prec h(z). \tag{9}$$

In order to do this we first introduce the function

$$L(z,t) \equiv h(z)\left[1+t\frac{\beta}{\gamma}\frac{zh'(z)}{h(z)}\right]^{1/\beta}, \tag{10}$$

where $t \geq 0$ and $z \in U$. We will show that $L(z, t)$ is a subordination chain by showing that it satisfies the conditions of Lemma 3. For $t \geq 0$ we have $|\arg(1 + \beta t/\gamma)| < \pi/2$, so if we set $w(t) = (1 + \beta t/\gamma)^{1/\beta}$ with $w(0) = 1$, we obtain $w(t) \neq 0$ and $\lim_{t\to\infty} |w(t)| = \infty$. Since $h(z) = b_1 z + \cdots$, with $b_1 \neq 0$, from (10) we obtain

$$L(z,t) = a_1(t)z + \cdots = b_1 w(t)z + \cdots.$$

Hence $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation using (10) yields

$$z\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t} = \gamma + (\beta - 1)\frac{tzh'(z)}{h(z)} + t\left[1+\frac{zh''(z)}{h(z)}\right].$$

Since $\beta > 1$, $t \geq 0$ and h is convex, we deduce that (4) is satisfied and, hence, $L(z, t)$ is a subordination chain. In particular, we have

$$h(z) = L(z,0) \prec L(z,t) \qquad \text{for all } t \geq 0. \tag{11}$$

We return now to the proof of (9). Suppose $F(z) \not\prec h(z)$. Then by Lemma 1 there exists points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that $F(z_0) = h(\zeta_0)$ and $z_0 F'(z_0) = m\zeta_0 h'(\zeta_0)$ with $m \geq 1$. Using this in (10), from (11) we obtain

$$F(z_0)\left[1+\frac{\beta}{\gamma}\frac{z_0F'(z_0)}{F(z_0)}\right]^{1/\beta} = L(\zeta_0, m) \notin h(U),$$

which contradicts the first part of (9). Hence $F(z) \prec h(z)$ and so $I_{\beta,\gamma}$ is an averaging operator on the set $\mathbf{E}_{\beta,\gamma}$.

Note that if f is also convex, then the conclusion of Theorem 1 becomes $I_{\beta,\gamma}[f] \prec f$.

If we let $f(z) = zg(z)$, $g(z) \neq 0$, then the integral in (7) can be rewritten as

$$z\left[\gamma \int_0^1 g^\beta(tz)t^{\beta+\gamma-1}\,dt\right]^{1/\beta}. \tag{12}$$

EXAMPLE 1. Let $f(z) = z/(1-z)$ and $g(z) = 1/(1-z)$. Then $f(z)$ is convex and from Theorem 1 and (12) we obtain

$$z\left[\gamma \int_0^1 (1-tz)^{-\beta}t^{\beta+\gamma-1}\,dt\right]^{1/\beta} \prec z/(1-z).$$

This last integral can be written in terms of the hypergeometric function

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\cdot\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$$

By taking $a = \beta$, $b = \beta + \gamma$, and $c = \beta + \gamma + 1$, we obtain $\Gamma(c)/[\Gamma(b)\cdot\Gamma(c-b)] = \beta + \gamma$ and deduce that

$$F(z) = z\left[\frac{\gamma}{\beta+\gamma}\,{}_2F_1(\beta,\beta+\gamma,\beta+\gamma+1;z)\right]^{1/\beta} \prec \frac{z}{1-z}$$

for $\beta \geq 1$ and $\operatorname{Re}\gamma > 0$. This also implies that $\operatorname{Re} F(z) > -\frac{1}{2}$.

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